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CLASSICAL PROPOSITIONAL CALCULUS

Symbols

Variables $P = \{p_n : n \in \omega\}$

Connectives: not, and, or --
 $\sim \quad \wedge \quad \vee$

Punctuation
Brackets $) , ($

String is a finite sequence
of symbols

Wff. is a 'meaningful' string.
 $F = \{Wff\}$

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A Realization is a map $f: F \rightarrow B_2$
2-element
Boolean Algebra

$$\forall \phi, \psi \in F$$

$$f(\phi \wedge \psi) = f(\phi) \wedge_2 f(\psi)$$

$$f(\neg \phi) = f(\phi)^*, \text{ etc}$$

where \wedge_2 is meet in B_2
* denotes complement in B_2

f satisfies ϕ (iff $f(\phi) = 1$ ($f \models \phi$))

ϕ is valid (on a tautology)

$$\text{iff. } \forall f (f \models \phi) \quad (\models \phi)$$

CHARACTERISTIC ALGEBRAS

Idea is to replace $f_{B_2} \in B_2^F$

by $f_A \in A^F$

Wff ϕ is satisfied by f_A iff

$$f_A(\phi) \in \mathcal{D}$$

\uparrow
set of designated
elements of A

ϕ is verified by A (i.e. is A -valid)

$$\text{iff } \forall f_A (f_A(\phi) \in \mathcal{D})$$

An algebra A is characteristic
for an abstract logical calculus
if theoremhood can be identified
with A -validity.

Theorem If B is an arbitrary Boolean Algebra a formula of CPC is B -Valid iff the formula is B_2 -Valid.

(1) B -Valid $\rightarrow B_2$ -Valid
 B always has B_2 as a sub-algebra.

(2) B_2 -Valid $\rightarrow B$ -Valid.
 Assume ϕ is B_2 -Valid but not B -Valid

So $\exists f_B$ s.t. $f_B(\phi) \neq 1$

Since $f_B(\phi) \neq 1$

$$f_B(\neg\phi) \neq 0$$

So by Ultrafilter Theorem

$f_B(\neg\phi)$ is contained in some
U/f of B

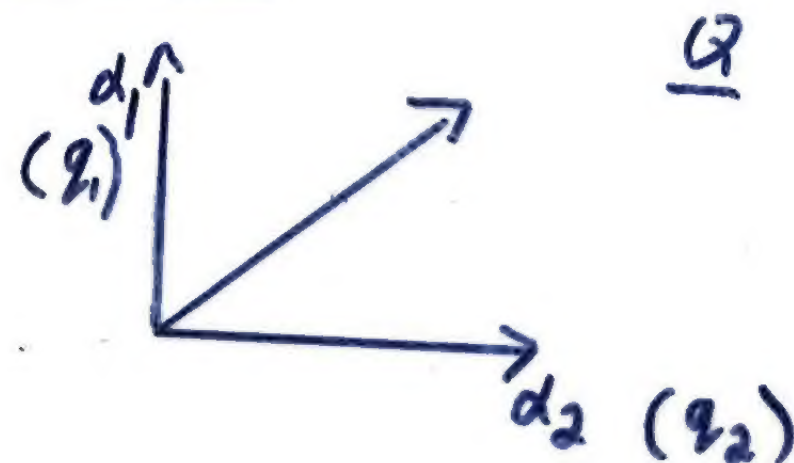
\therefore by Homomorphism Theorem

$f_B(\neg\phi) \mapsto 1$ under some
homomorphism $H: B \rightarrow B_2$

$\therefore f_B(\phi) \mapsto 0$ which implies
a B_2 -realization $f = H \circ f_B$
for which $f(\phi) = 0$, contrary
to hypothesis - Result follows
by Reductio.

CONNECTIVES IN QM

DISTUNCTION:



P_1 : state of system is associated with q_1 or d_1

P_2 : d_2

P_1 or P_2 : state of system in which result of measurement is q_1 or q_2

\Rightarrow state-vector lies in linear span of d_1, d_2 (as in CM)

QUANTUM NEGATION

$\neg P$: state is such that result
of measurement never yields q

Compare classical negation:

$\neg P$: state is such that result
of measurement is not always q

QUANTUM CONJUNCTION

$P_1 \& P_2 =_{\text{df}} \neg(\neg P_1 \text{ or } \neg P_2)$

Ex P_1 says q yields q_1 or q_2 on measurement

P_2 q_2 or q_3 . . .

$P_1 \& P_2$ q_2 only . . .

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Proot that $\text{val}[\{1\}]_{P_Q(\{2\})}^\phi = \text{val}[\{2\}]_Q^\phi$

An observable Q is a map $Q: \{\Delta\} \rightarrow \{P\}$

$$\Delta \mapsto P_Q(\Delta)$$

Now $\text{Proot}[\Delta]_{f(Q)}^\phi = \text{Proot}[f^{-1}(\Delta)]_Q^\phi$

Assume (functional composition principle)

$$\text{val}[\Delta]_{f(Q)}^\phi = \text{val}[f^{-1}(\Delta)]_Q^\phi$$

Take $f(Q) = \chi_\Delta(Q) = P_Q(\Delta)$

Then $\text{val}[\{1\}]_{P_Q(\Delta)}^\phi = \text{val}[\chi_\Delta^{-1}\{1\}]_Q^\phi$
 $= \text{val}[\Delta]_Q^\phi$

Result follows if we take $\Delta = \{2\}$

THE TWO-COLOUR THEOREM

On a hypersphere in a Euclidean space of three or more dimensions it is not possible to colour every point with either of 2 colours, red & blue, so that for every orthogonal N -tuple of points only one is coloured red.